- The Fourier series is an extremely useful signal representation.
- Unfortunately, this signal representation can only be used for periodic signals, since a Fourier series is inherently periodic.
- Many signals are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can be applied to aperiodic signals.
- By viewing an aperiodic signal as the limiting case of a periodic signal with period T where $T \to \infty$, we can use the Fourier series to develop a more general signal representation that can be used for both aperiodic and periodic signals.
- This more general signal representation is called the Fourier transform.

- The (CT) Fourier transform of the signal x, denoted $\mathcal{F}\{x\}$ or X, is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

- The preceding equation is sometimes referred to as Fourier transform analysis equation (or forward Fourier transform equation.(
- The inverse Fourier transform of X, denoted $\mathcal{F}^{-1}\{X\}$ or x, is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

- The preceding equation is sometimes referred to as the Fourier transform synthesis equation (or inverse Fourier transform equation). As a matter
- of notation, to denote that a signal x has the Fourier transform X, we write $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$.
- A signal x and its Fourier transform X constitute what is called a Fourier transform pair.

Section 5.2

Convergence Properties of the Fourier Transform

- Consider an arbitrary signal x.
- The signal x has the Fourier transform representation $\tilde{x}$ given by

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

- Now, we need to concern ourselves with the convergence properties of this representation.
- In other words, we want to know when $\tilde{x}$ is a valid representation of x.
- Since the Fourier transform is essentially derived from Fourier series, the convergence properties of the Fourier transform are closely related to the convergence properties of Fourier series.

- If a signal x is *continuous* and *absolutely integrable* (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$) and the Fourier transform X of x is absolutely integrable (i.e., $\int_{-\infty}^{\infty} |X(\omega)|\, d\omega < \infty$), then the Fourier transform representation of x converges *pointwise* (i.e., $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] e^{j\omega t} d\omega$ for all t).
- Since, in practice, we often encounter signals with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value.

- If a signal x is of *finite energy* (i.e., $\int_{-\infty}^{\infty} |x(t)|^2\,dt < \infty$), then its Fourier transform representation converges in the *MSE sense*.
- In other words, if x is of finite energy, then the energy E in the difference signal $\tilde{x} - x$ is zero; that is,

$$E = \int_{-\infty}^{\infty} |\tilde{x}(t) - x(t)|^2\,dt = 0.$$

- Since, in situations of practice interest, the finite-energy condition in the above theorem is often satisfied, the theorem is frequently applicable.
- It is important to note, however, that the condition $E = 0$ does not necessarily imply $\tilde{x}(t) = x(t)$ for all t.
- Thus, the above convergence result does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier transform representation converges.

- The Dirichlet conditions for the signal x are as follows:
  1. The signal x is *absolutely integrable* (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$).
  2. On any finite interval, x has a finite number of maxima and minima (i.e., x is of *bounded variation*).
  3. On any finite interval, x has a *finite number of discontinuities* and each discontinuity is itself *finite*.
- If a signal x satisfies the *Dirichlet conditions*, then:
  1. The Fourier transform representation $\tilde{x}$ converges pointwise everywhere to x, except at the points of discontinuity of x.
  2. At each point $t = t_a$ of discontinuity, the Fourier transform representation $\tilde{x}$ converges to

$$\tilde{x}(t_a) = \tfrac{1}{2}\left[x(t_a^+) + x(t_a^-)\right],$$

where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal x on the left- and right-hand sides of the discontinuity, respectively.

- Since most signals tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier transform representation at every point, this result is often very useful in practice.

Section 5.3

Properties of the Fourier Transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting Frequency-	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Domain Shifting Time/Frequency-	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Domain Scaling Conjugation	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Duality	$x^*(t)$	$X^*(-\omega)$
Time-Domain Convolution	$X(t)$	$2\pi x(-\omega)$
Frequency-Domain Convolution	$x_1*x_2(t)$	$X_1(\omega)X_2(\omega)$
Time-Domain Differentiation	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1*X_2(\omega)$
Frequency-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2\,d\omega$

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}(t/T)$	$\lvert T\rvert \operatorname{sinc}(T\omega/2)$
9	$\frac{\lvert B\rvert}{\pi}\operatorname{sinc} Bt$	$\operatorname{rect}\frac{\omega}{2B}$
10	$e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{\lvert T\rvert}{2}\operatorname{sinc}^2(T\omega/4)$

- If $x_1(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$a_1 x_1(t) + a_2 x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} a_1 X_1(\omega) + a_2 X_2(\omega),$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the linearity property of the Fourier transform.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(t-t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

  where t_0 is an arbitrary real constant.
- This is known as the translation (or time-domain shifting) property of the Fourier transform.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$e^{j\omega_0 t}x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega-\omega_0),$$

where ω_0 is an arbitrary real constant.

- This is known as the modulation (or frequency-domain shifting) property of the Fourier transform.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant.

- This is known as the dilation (or time/frequency-scaling) property of the Fourier transform.

- If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$x^*(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X^*(-\omega.($$
- This is known as the conjugation property of the Fourier transform.